Situation 51: Proof by Mathematical Induction
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Prompt

A teacher of a calculus course gave her students the opportunity to earn extra credit by proving various algebraic formulas by mathematical induction. For example, one of the formulas was the following:

$$1^2 + 2^2 + ... + n^2 = (n)(n + 1)(2n + 1)/6.$$

Only one student in the class was able to prove any of the formulas. After the student had presented his three proofs to the class on three consecutive days, another student complained: "I don't get what he is proving. And besides that, how do you get the algebraic formulas to start with?"

Commentary

Given that it is a deductive process, *mathematical induction* is an unfortunate term. George Pólya used to say that a better term would be *mathematical complement to induction*. Although an inductive process can be used to verify a conjecture involving *n* for any finite *n,* mathematical induction is needed to prove the conjecture for all natural numbers *n.*

The foci consider several important aspects of developing proofs by mathematical induction that students might find difficult or confusing, as well as some ways of generating conjectures that have an inductive proof. Focus 1 looks at a common image for what proof by mathematical induction accomplishes. Focus 2 examines a problem context in which a recursion relationship might be formed and used to formulate an algebraic statement that can be proved by mathematical induction. Focus 3 gives several examples of how figurate representation of numbers can lead to conjectures about algebraic formulas and subsequent proofs by induction. Focus 4 analyzes the structure of inductive proof by providing some examples in which one of the two conditions for induction fails. It suggests why a problem situation might be a better starting point for a proof by mathematical induction than an algebraic formula (true or false) because in the latter case, one starts with an analysis of a type of proof rather than the generation of a proof itself.

Mathematical Foci

Mathematical Focus 1
*Proof by mathematical induction can be portrayed with an image of any process involving a basis (*k = 0*) and an inductive step (if the conjecture holds for* n = k, *it holds for* n = k + 1*).*

It may be useful to provide some images of what mathematical induction does. One good image might be of a ladder that is infinitely tall (to represent the integers). In doing a proof by induction, we first have to check whether we can get on the first rung of the ladder (which corresponds to checking whether the base case of the proposition holds true). Once we know we can get on the ladder, we have to check that if we are on any step of the ladder, we can proceed to the next one; namely, if we are on the kth rung, we can move to the $(k + 1)$st rung (which corresponds to showing that if the proposition holds for k it will hold for $k + 1$). By proving these two things, we claim that, using the principle of mathematical induction, we can get to any rung of the ladder (which corresponds to proving that the proposition is true for all k).

An alternate image used to illustrate proof by mathematical induction involves the domino effect. Imagine an infinite row of dominoes standing on end. If one knows that the first domino will fall, and if one knows that when domino k falls, its next neighbor $(k + 1)$ will fall, then by the principle of mathematical induction, all the dominos will fall.

Mathematical Focus 2
Proof by mathematical induction can be motivated by a recursion relationship using a graphic display.

Many recursion formulas create problem-solving opportunities that might lead to an algebraic formulation of a situation and a subsequent proof by mathematical induction. One such problem situation[1] is the following:

Suppose you have a $2^n \times 2^n$ chessboard. Show that if you remove one square (shown in black), you can cover the board with L-triominoes (shown in purple).
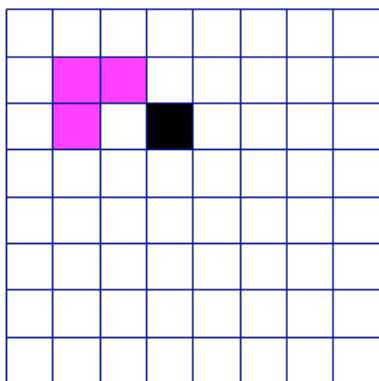

In the case shown above, there is an 8×8 chessboard. It is easy to show that it is possible to find a covering of the chessboard, but it is a little difficult to see a pattern in a particular covering of a chessboard of this size. This difficulty provides a good opportunity to use Polya's heuristic strategy of looking at a related problem that is simpler. Let's examine a 2×2 and 4×4 chessboard. In the 2×2 case, we can clearly cover the chessboard without any difficulty. That is, we can place a triomino as we please, and exactly one space will be left over:

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[1] This problem situation arose in Patricia Wilson's EMAT 6650 (Historical and Cultural Foundations of Mathematics) course.

We might next look at a 4 × 4 board and notice that it is composed of exactly four 2 × 2 chessboards (and further that there are four 4 × 4 chessboards in an 8 × 8 chessboard). In making this observation, we can establish a recursion relationship between successive chessboards. Also, we might observe that our 4 × 4 board will have exactly one 2 × 2 chessboard with the space taken out of it and three 2 × 2 chessboards with exactly one space not covered. We can cover those three spaces with exactly one triomino (shown on the left).
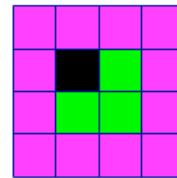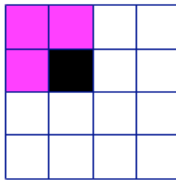
Having made these observations, we can set up a recursion relationship between successive boards and make a more formal proof by mathematical induction. Note that in this case we might simply do the proof by writing out our observations.

On the other hand, we might use this problem to show that $(2^{2n} - 1)/3$ is a whole number for all $n > 1$. To think of this algebraic formulation, we notice that the chessboard contains 2^{2n} squares and that removing one of the squares gives us $2^{2n} - 1$ remaining squares to be covered. Also, each triomino contains 3 squares. Most importantly, we have to see that the spatial arrangement of the triominoes on the board does not affect the solution of the problem. Imagine cutting the board at each row and putting the rows end to end. Then imagine changing the L-triominoes to straight triominoes (shown below for the 4 × 4 chessboard).
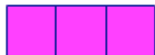
We can once again use a recursive argument to show that we can cover all but one of the squares of our transformed board with our transformed triominoes. From this spatial configuration, we may become more convinced that we are asking ourselves a question of division. That is, how many threes are in a row of length $2^{2n} - 1$? The answer is a whole number.

Mathematical Focus 3
Figurate numbers can also be used to generate identities that can then be proved by mathematical induction.

Consider a figurate representation for the square numbers using dots. In that configuration, one might observe the pattern below:
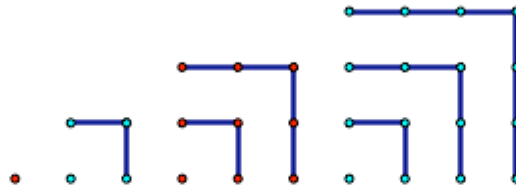
The configuration suggests that $n^2 = 1 + 3 + \ldots + (2n - 1)$. Having made a conjecture about the formula, we can then try to prove it by mathematical induction. (Also, how might a formula for the sum of the first n even numbers be found and proved?)

A second situation that uses figurate numbers and that extends Situation 38, which demonstrates several ways to get an intuitive feeling for why the sum of the first n integers equals $\dfrac{n(n+1)}{2}$, is to find the sum of the first n sums of the first n integers. Algebraically, if $T_1 = 1$, $T_2 = 1 + 2$, … $T_n = 1 + 2 + \ldots + n$, then we want to find $T_1 + T_2 + \ldots + T_n$. We can introduce the situation by finding both an additive and multiplicative solution to the following problem. Suppose there are five points on a circle. How many possible triangles can you make?
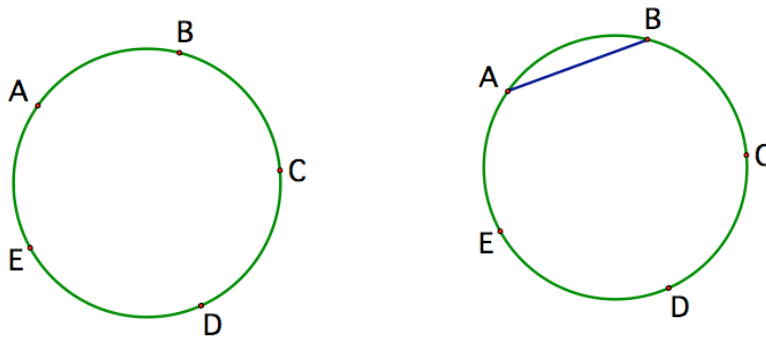

We can structure the problem additively by thinking about the number of triangles with AB as the base (3), with BC as the base (2 more), and with CD as the base (1 more). Multiplicatively, we can think about choosing sets of three points from five or $\binom{5}{3}$. After solving the problem for several numbers of points, we might conjecture that

$$T_1 + T_2 + \ldots + T_n = \binom{n+2}{3} = \tfrac{1}{6}(n+2)(n+1)(n),$$ and use the problem context to prove the identity

by mathematical induction. (An extension of the problem would be to find an additive and multiplicative way of thinking about the number of diagonals that can be drawn in a convex n-gon).

We might then consider using the identity to find an identity for $1^2 + 2^2 + ... + n^2$. We again make a figurate representation of the problem to help us see what the identity might be. The first four square numbers are shown below:
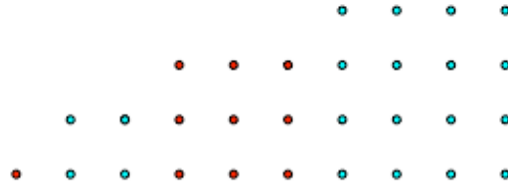

Knowing that we have a formula for the sum of the first n triangular numbers, we break the figure into triangular numbers:
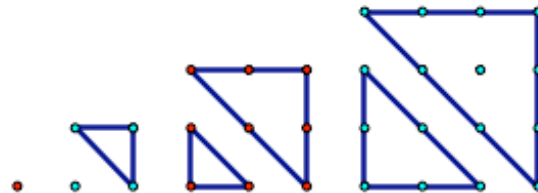
From our example of the sum of the first four square numbers, we might observe that the sum of the first n square numbers is $2T_1 + 2T_2 + ... + 2T_{n-1} + T_n = \binom{n+2}{3} + \binom{n+1}{3}$. A little algebraic manipulation gives $1^2 + 2^2 + ...n^2 = \frac{1}{6}(2n+1)(n+1)(n)$, another formula that we conjecture to be true and whose validity might be established by mathematical induction.

A final problem might be to look at the square of the sum of the first n integers: $(1 + 2 + ... + n)^2$. We again might want to make a figurate representation of the case for $n = 4$:
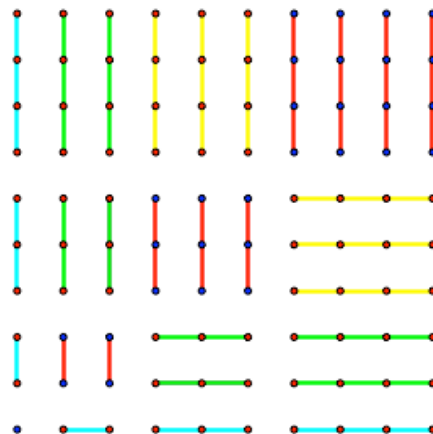
How many groups of four are in the figure above? There are $\binom{5}{2}$ vertical groups (the sum of the first four integers), and there are $\binom{4}{2}$ horizontal groups (the sum of the first three integers), giving a total of $4(\binom{5}{2} + \binom{4}{2})$ dots. A little algebraic simplification shows that there are 4^3 dots in the outermost region of the figure. A similar process shows us that in computing the dots in groups of threes gives us 3^3 dots, and so on. That yields a conjecture for the following identity: $(1 + 2 +\ldots n)^2 = 1^3 + 2^3 +\ldots + n^3$. We then can prove the formula by mathematical induction.

Mathematical Focus 4
Proof by mathematical induction requires both a basis step and an inductive step; it fails if either is lacking.

Many of the examples in which one of the steps fails are somewhat trivial, but they can nonetheless give us some feeling for the importance of both.

For instance, consider the inequality $5(2^n) > 6^n$. It holds true for $n = 1$ (we can get on the ladder). Now assume the inequality is true for $n = k$ and try to show that it is true for $n = k + 1$. We need to show that $5(2^{k+1}) > 6^{k+1}$ given that $5(2^k) > 6^k$. We use the inductive hypothesis to form the inequality $5(2^{k+1}) = 2(5(2^k)) > 2(6^k)$, and we would like to be able to conclude that $2(6^k) > 6(6^k) = 6^{k+1}$. However, that would imply that $2 > 6$, which is false. Seeing no way to proceed, we check our assumption. If $n = 2$, $5(2^n) > 6^n$ yields $20 > 36$, which is false. In this case, we were able to get on the first step of the ladder but not able to move from rung to rung.

We might now consider the inequality $3^n > 5(3^n)$. Suppose we begin by assuming that it is true for $n = k$ and try to prove it for $n = k + 1$. We find that $3^{k+1} = 3(3^k) > 3(5(3^k)) = 5(3^{k+1})$, and the inequality holds true based on our inductive hypothesis (we are able to move from rung to rung). However, in the case of $n = 1$ the inequality does not hold. We are unable to get on the ladder even though if we could, we could move from rung to rung. These examples illustrate that both conditions need to be met in for a proof by mathematical induction to work. They also suggest that it is worth checking *several* cases when proving something by induction.

Checking several cases, however, is not a substitute for a proof, as can be seen by the inequality $3^n > n!$. This inequality holds for the first six cases, but it is false when $n = 7$. For any a, the inequality $a^n > n!$ will eventually become false even though we can make the n for which it becomes false arbitrarily large. This example demonstrates that proof by mathematical induction is in fact deductive. That is, we are not drawing a conclusion based solely on repeated observations (inductive reasoning), instead, we are establishing two premises and stating that if both premises hold true we can assert the conclusion (deductive reasoning). The three examples in this focus taken together suggest the relationship of the premises to the conclusion as well as the role that observation plays in relation to a proof by induction.